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Quonic Realization of the Deformed Heisenberg–Weyl Algebra

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A consistent realization of an operatorial deformed Heisenberg–Weyl (HW) algebra is given. The explicit construction of the deformation matrix is discussed. The corresponding generalized Fock-like space is analyzed.

In the last few years, great interest has been devoted to the study and the understanding of intermediate statistics, also called exotic statistics (Leineas and Myrheim, 1977). The latter describe particles interpolating between bosons and fermions and are characterized by a fractional spin or more generally by fractional quantum numbers. These exotic particles appear in the study of the fractional quantum Hall effect and in the theory of high-temperature superconductivity (Halperin, 1984; Chen *et al.*, 1989). A possibility to shed some light on caracteristic symmetries of these particular systems of particles involves quantum groups which are deformations of ordinary Lie algebras. It has been shown (Biedenharn, 1989; Macfarlane, 1989) that the realization of these mathematical objects can be obtained by using some consistent q-deformation of the oscillator algebras.

Recently, it was remarked (De Falco *et al.*, 1995; De Falco and Mignani, 1996) that the deformed Heisenberg–Weyl (HW) algebra appears in the intermediate statistics. To realize this (HW) algebra the authors consider some deformation matrix G instead of just a deformation parameter q.

Indeed, an explicit calculus concerning a bosonic and a fermionic even-dimensional realization of an operatorial deformation of the HW algebra has been performed. In the present work, we give a generalization of this study, and by using a mathematical framework of quonic algebra, we obtain a

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consistent deformation of the HW algebra. We prove that this algebra is realized starting from particles having different fractional spins. We discuss a corresponding Fock-like space and also show how to recover the bosonic and fermionic Fock spaces for some particular values of the corresponding deformation matrix.

Let us first recall the quonic algebra defined (Macfarlane, 1989) through the three elements a_i, $a_i^\dagger$, and N_i satisfying

$$a_i a_i^\dagger - q_i a_i^\dagger a_i = 1 \tag{1a}$$

$$[N_i, a_i^\dagger] = a_i^\dagger \tag{1b}$$

$$[N_i, a_i] = -a_i \tag{1c}$$

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0 \tag{1d}$$

with $i = 1, 2 \ldots$, and where the parameter q_i is a root of unity ($q^{k_i} = 1$; $k_i \geq 2$). The corresponding Fock-like space is given by

$$F_j = \{|n_j\rangle, n_j = 0, \ldots, k_j - 1\} \tag{2}$$

The deformation parameter is related to the spin of the above exotic particles; $s_i = 1/k_i$. We point out that, for any oscillator, the generalized Pauli exclusion principle is assumed. In fact, one suggests that no more than $(k_j - 1)$ identical particles can live in the same quantum state. The action of a_j, $a_j^\dagger$, and N_j on F_j is given by

$$\begin{aligned} a_j^\dagger|n_j\rangle &= |n_j + 1\rangle \\ a_j|n_j\rangle &= [n_j]|n_j - 1\rangle \\ N_j|n_j\rangle &= n_j|n_j\rangle \end{aligned} \tag{3}$$

with

$$[x] = \frac{q^x - 1}{q - 1} \tag{3'}$$

Moreover, the operators a_j and $a_j^\dagger$ satisfy the nilpotency conditions

$$(a_j)^{k_i} = (a_j^\dagger)^{k_i} = 0 \tag{4}$$

These relations indicate that quons obey the generalized Pauli exclusion principle.

We note also that the quonic annihilator and creator satisfy the interesting polynomial relation

$$a_j^{k_j-1}a_j^\dagger + a_j^{k_j-2}a_j^\dagger a_i + \cdots + a_j^\dagger a_j^{k_j-1} = \frac{k_i}{1-q_j}a_j^{k_j-2} \tag{5}$$

This relation allows us to realize fractional supersymmetric quantum mechanics (Daoud and Hassouni, n.d.).

Using these tools, we define the Fock-like space constructed from the direct sum of quonic Fock spaces F_j defined in equalities (2). This space is given by

$$F = \bigoplus_{j=0}^{\alpha-1} F_j, \qquad \alpha \in N^* \tag{6}$$

Any vector space F_j can be seen as a quonic Fock space, and its dimension is k_j.

Now, let us define the annihilation and creation operators acting on F; we have

$$A^\dagger = \bigoplus_{j=0}^{\alpha-1} a_j^\dagger \tag{7a}$$

$$A = \bigoplus_{j=0}^{\alpha-1} a \tag{7b}$$

We assume that the vacuum state is described by

$$|0\rangle \equiv \begin{pmatrix} |0\rangle_0 \\ |0\rangle_1 \\ \vdots \\ |0\rangle_{\alpha-1} \end{pmatrix} \tag{8}$$

and satisfies $A|0\rangle = 0$. The one-particle state is $A^\dagger|0\rangle$ and generally the n-particles state is of the form $(A^\dagger)^n|0\rangle$:

$$A^\dagger|0\rangle \equiv \begin{pmatrix} a_0^\dagger|0\rangle_0 \\ a_1^\dagger|0\rangle_1 \\ \vdots \\ a_{\alpha-1}^\dagger|0\rangle_{\alpha-1} \end{pmatrix} \equiv |1\rangle \tag{9}$$

$$(A^\dagger)^n|0\rangle \equiv \begin{pmatrix} (a_0^\dagger)^n|0\rangle_0 \\ (a_1^\dagger)^n|0\rangle_1 \\ \vdots \\ (a_{\alpha-1}^\dagger)^n|0\rangle_{\alpha-1} \end{pmatrix} \equiv |n\rangle \tag{10}$$

Let us introduce the corresponding algebra and the commutation relation between their elements. First, we define the generalized number operator given by

$$
[N]_G \equiv \begin{pmatrix} [N_0]_{q^0} & & & & \\ & \ddots & & & \\ & & [N_j]_{q^j} & & \\ & & & \ddots & \\ & & & & [N_{\alpha-1}]_{q^{\alpha-1}} \end{pmatrix}
= \begin{pmatrix} a_0^\dagger a_0 & & & & \\ & \ddots & & & \\ & & a_j^\dagger a_j & & \\ & & & \ddots & \\ & & & & a_{\alpha-1}^\dagger a_{\alpha-1} \end{pmatrix} \tag{11}
$$

In this study, we have considered q^α-deformed oscillators. The oscillator (i) is q^i-deformed. Using the definition of annihilation, creation, and generalized number operators, one can verify that the G-(HW) algebra generated by A, $A^\dagger$, and N satisfies the following commutation relation:

$$
[A, A^\dagger]_G \equiv AA^\dagger - GA^\dagger A = \mathbf{1}
$$
$$
[N, A^\dagger] = A^\dagger, \qquad [N, A] = -A \tag{12}
$$

where

$$
AA^\dagger = \begin{pmatrix} a_0 a_0^\dagger & & & & \\ & \ddots & & & \\ & & a_i a_i^\dagger & & \\ & & & \ddots & \\ & & & & a_{\alpha-1} a_{\alpha-1}^\dagger \end{pmatrix} \tag{13a}
$$

$$
A^\dagger A = \begin{pmatrix} a_0^\dagger a_0 & & & & \\ & \ddots & & & \\ & & a_i^\dagger a_i & & \\ & & & \ddots & \\ & & & & a_{\alpha-1}^\dagger a_{\alpha-1} \end{pmatrix} \tag{13b}
$$

The deformation matrix is G given by

$$
G = \begin{pmatrix} 1 & & & & \\ & q & & & \\ & & q^2 & & \\ & & & \ddots & \\ & & & & q^{\alpha-1} \end{pmatrix} \tag{14}
$$

with

$$q = \exp(2\pi i/l) \tag{15}$$

and

$$l = \alpha! \tag{16}$$

In this case the dimension of a given Fock space F_j is equal to l/j ($j = 1, \ldots, \alpha - 1$). The Fock space F_0 is infinite-dimensional (bosonic one). The choice of $l = \alpha!$ is necessary in the sense that the dimension of any F_j ($j \in N^*$) must be integer. This choice leads to the interesting relation

$$G^l = \mathbf{1} \tag{17}$$

To end our study, we will discuss the case $l = 2$ ($\alpha = 0, 1$). Thus, the corresponding Fock-like space is given by

$$F = F_0 \oplus F_1 \tag{18}$$

The creation and the annihilation operators are given respectively by

$$A^\dagger = \begin{pmatrix} a_0^\dagger & 0 \\ 0 & a_1^\dagger \end{pmatrix}, \qquad A = \begin{pmatrix} a_0 & 0 \\ 0 & a_1 \end{pmatrix} \tag{19}$$

The deformation matrix is

$$G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{20}$$

which satisfies the condition $G^2 = \mathbf{1}$, and the number operator is

$$N = \begin{pmatrix} N_0 & 0 \\ 0 & N_1 \end{pmatrix} \tag{21}$$

It is clear that one can recover the results obtained in (De Falco *et al.*, 1995) and (De Falco and Mignani, 1996) from our construction, in the particular case $l = 2$ ($\alpha = 0, 1$).

The algebra generated by a_0, $a_0^\dagger$, and N_0 is bosonic:

$$\begin{aligned} [a_0, a_0^\dagger] &= a_0 a_0^\dagger - a_0^\dagger a_0 = \mathbf{1} \\ [N_0, a_0] &= -a_0 \\ [N_0, a_0^\dagger] &= a_0^\dagger \end{aligned} \tag{22}$$

and for the oscillator specified by a_1, $a_1^\dagger$, and N_1, we have the fermionic algebra:

$$\begin{aligned} \{a_1, a_1^\dagger\} &= a_1 a_1^\dagger + a_1^\dagger a_1 = \mathbf{1} \\ [N_1, a_1] &= -a_1 \\ [N_1, a_1^\dagger] &= a_1^\dagger \end{aligned} \tag{23}$$

we remark that by taking $l = 2$, we recover a generalized Fock-like space described by bosons and fermions. This is the result obtained in (De Falco *et al.*, 1995; De Falco and Mignani, 1996) corresponding to the Fock-like space $F = F_1 \oplus F_0$, which we call the fermionic realization.

In the latter case one can consider the operators $B = a_1 \oplus a_0$ and $B^\dagger = a_1^\dagger \oplus a_0^\dagger$. The matrix realization is then given by

$$B = \begin{pmatrix} a_1 & 0 \\ 0 & a_0 \end{pmatrix} \tag{24a}$$

$$B^\dagger = \begin{pmatrix} a_1^\dagger & 0 \\ 0 & a_0^\dagger \end{pmatrix} \tag{24b}$$

Thus the number operator is given by

$$N' = \begin{pmatrix} N_1 & 0 \\ 0 & N_0 \end{pmatrix} \tag{25}$$

Instead of the realization given by (12), one have to consider the following commutation relations:

$$\begin{aligned} \{B, B^\dagger\}_G &= BB^\dagger + GB^\dagger B = 1 \\ [N', B] = -B, &\qquad [N', B^\dagger] = B^\dagger \end{aligned} \tag{26}$$

with

$$G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{27}$$

The algebras given by A, $A^\dagger$, and N thus obtained by B, $B^\dagger$, and N' are equivalent in the sense that there exists a transformation U such that

$$\begin{aligned} UAU^\dagger &= B \\ UA^\dagger U^\dagger &= B^\dagger \\ UGU^\dagger &= -G \end{aligned} \tag{28}$$

with

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{29}$$

This transformation becomes not interesting for the case $l > 2$. Indeed, in this case, we have a quonic realization, and seeing that the quon interpolates between the boson and the fermion, one cannot imagine an analog of the above U transformation for $l > 2$.

In summary, this work has been devoted to the realization of an algebraic structure of the deformed HW algebra. This generalization is obtained by considering a deformation operator instead of just a deformation parameter. The deformation operator is constructed starting from a set of quonic oscillators. It is very interesting to investigate the possibility of application of our results in the context of anyonic statistics. More precisely, we think that they can be related to results given in (Daoud and Hassouni, 1997). We intend to study the fractional supersymmetry starting from this realization. We would like to construct fractional supersymmetric charges $Q^\dagger$ and Q (with $Q^{\dagger l} = Q^l = 0$) using the G-(HW) algebra introduced above (Daoud *et al.*, n.d.).

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